The inductance per unit length is given by

$$L = \frac{\Phi}{lI} \tag{8}$$

where

$$\Phi = \int_A \boldsymbol{B}\cdot d\boldsymbol{A} \quad \text{(Gaussian Law)} \tag{9}$$

and

$$I = \oint \boldsymbol{H}\cdot d\boldsymbol{s} \quad \text{(Ampere's Law).} \tag{10}$$

Consider now a closed loop of a typical line of magnetic field around the center conductor of a slab line (Fig. 2). If such a loop is approximated by a rectangular one of about the same length, $2W+2T+4y$, the magnetic field intensity H at distance y from the center conductor can be calculated from (10) and is found to be

$$H = \frac{I}{2W+2T+4y}\cdot \tag{11}$$

Substituting for the magnetic flux density, $B(=\mu H)$ in (9), changing the variable of integration from A to y with $dA = l dy$, putting in appropriate limits of integration and rearranging terms,

$$\frac{\Phi}{lI} = L = \int_0^{(D-T)/2} \frac{\mu dy}{2W+2T+4y} \tag{12}$$

from which it readily follows that

$$L = \frac{\mu}{4}\ln\left[\frac{W+D}{W+T}\right]. \tag{13}$$

Substituting for L in (7), putting in the values for c_0' and μ, (6) is obtained.

A graphical comparison of (5) and (6) is shown in Fig. 3 where the fractional error in characteristic impedance is plotted as a function of W/D for $T/D = 0.1$. For $W/D < 1.0$ the error in calculating Z_0 using (6) becomes increasingly greater than 1 percent. However, its range of application can be extended by considering a longer line of magnetic field, of length $2KW+2T+4y$, where $K>1$ and is a function of W/D. Equation (6) modifies to

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_r}}\ln\left(\frac{KW+D}{KW+T}\right). \tag{14}$$

By using (5) and (14), values of K as a function of W/D have been found and plotted graphically (Fig. 4), which allow (14) to be used to determine Z_0 to within 0.7 percent of Cohn's value, for $W/D \geq 0.30$, and $T/D \leq 0.10$. For $T/D \leq 0.20$, Z_0 can be determined to within about 5 percent using (14) for $W/D \geq 0.30$.

Finally, for the special case of zero strip thickness ($T/D = 0$), (5) simplifies to the well-known expression

$$Z_0 = \frac{94.15/\sqrt{\epsilon_r}}{\left[\frac{W}{D}+\frac{\ln 4}{\pi}\right]}\cdot \tag{15}$$

Similarly, (6) reduces to

$$Z_0 = \frac{30\pi}{\sqrt{\epsilon_r}}\ln\left[1+\frac{D}{W}\right] \tag{16}$$

which is accurate to within almost 1 percent of (15) for $W/D \geq 0.75$.

Acknowledgment

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An Improved Method for Measuring Scattering Parameters of Nonreciprocal Two-Ports

Usual methods for measuring the characteristic parameters of reciprocal structures fail when applied to nonreciprocal structures. Methods which rest on the use of the input impedance formula

$$Z_{in} = \frac{(Z_{11}Z_{22} - Z_{12}Z_{21}) + Z_{11}Z_{out}}{Z_{22} + Z_{out}}$$

or, of the input reflection coefficient relationship

$$K_{in} = S_{11} - \frac{S_{12}S_{21}}{S_{22} - \frac{1}{K_{out}}}$$

allow determination of the products $Z_{12}Z_{21}$ or $S_{12}S_{21}$, but are unable to separate the individual factors. Another difficulty arises when one considers that practical magnitudes of the scattering parameters often lie very close to zero or unity (e.g., for isolators or for phase shifters). The method which we propose allows accurate determination of both magnitude and phase of the scattering parameters. The procedure gives good results for arbitrary values of the parameters, and is an extension of a procedure suggested by Macpherson [1] and Pippin [2], and adapted by Altschuler [3].

Consider a nonreciprocal two-port with reference planes 1 and 2 (Fig. 1). Then,

$$b_1 = S_{11}a_1 + S_{12}a_2 \tag{1}$$

$$b_2 = S_{21}a_1 + S_{22}a_2. \tag{2}$$

Let us assume that $|a_1| = |a_2|$ and $a_1/a_2 = ej\theta$. The reflection coefficient at the reference

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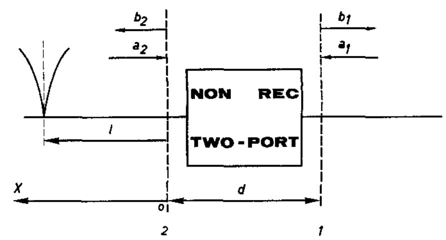

Fig. 1. Nonreciprocal two-port with its reference indications.

plane 2, K_2, then takes the value, from (2),

$$K_2 = \frac{b_2}{a_2} = S_{22} + S_{21}e^{j\theta}. \tag{3}$$

By means of a calibrated phase shifter at one side of the two-port, θ can be adjusted to take the values 0 or π. This yields

$$(K_2)_0 = S_{22} + S_{21} \tag{4}$$

$$(K_2)_\pi = S_{22} - S_{21}. \quad \text{(Pippin)} \tag{5}$$

From (4) and (5) S_{22} and S_{21} can be readily computed, but with the poor accuracy attached to a one point method. Altschuler improved the situation by remarking that, when θ varies from 0 to 2 π, K_2 describes a circle in the complex plane. The center of the circle corresponds to S_{22} and the radius to $|S_{21}|$. The use of a large number of points now allows averaging of the experimental errors. With practical structures, however, this method still is not satisfactory. If S_{22} and S_{21} differ too much, for example, it is impossible to locate the center of the circle with accuracy. Furthermore, high VSWR must be measured, which is both difficult and inaccurate. If S_{22} and S_{21} are both very close to zero, very low VSWR must be measured, and the precision drops again because of the mismatch errors. Accurate phase determination is still difficult because it depends on a one point measurement. To improve the method, let us consider the equation

$$K_2 = S_{22} + S_{21}\frac{a_1}{a_2}, \tag{6}$$

which can be rewritten as

$$K_2 = S_{22} + S_{21}\left|\frac{a_1}{a_2}\right| e^{j(\arg a_1 - \arg a_2)}. \tag{7}$$

The radius of the circle is now $|S_{21}|\cdot|a_1/a_2|$. If we succeed in adjusting $|a_1/a_2|$ to any desired value, the original circle of Altschuler will be compressed or expanded and the measurements become both easy and accurate. Besides, really accurate measurements can be performed by utilizing several values of $|a_1/a_2|$, i.e., by plotting several circles in the complex plane. From each circle, a value $|S_{21}|$ can be obtained and the various values can be averaged. This procedure smooths out the errors of the attenuators. Points with constant phase θ must lie on straight lines through the common center of the circles. Phase determination, therefore, becomes accurate too (Fig. 2). The practical set-up is shown in Fig. 3. The matched pads consist of an isolator matched with an E-H tuner to avoid multiple reflections. The measurements proceed as follows:

1) The structure under test is replaced by a piece of waveguide with the same

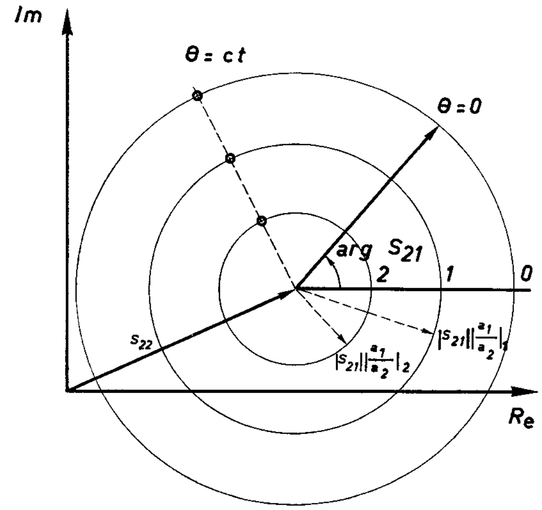

Fig. 2. Typical plot of (7).

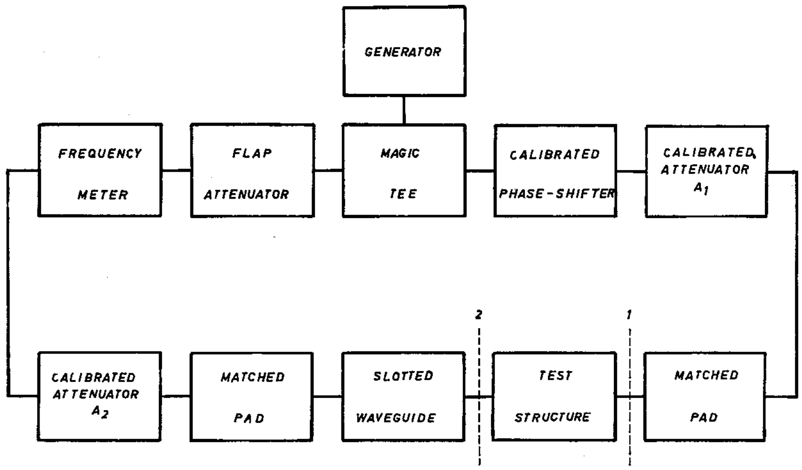

Fig. 3. Experimental setup.

physical length d. A_1 and A_2 are set for zero attenuation and the flap attenuator is adjusted until a short-circuit pattern is seen in the slotted waveguide. This means that $|a_1| = |a_2|$. The location of the minima is noted. By adjusting A_1 or A_2, any value of $|a_1/a_2|$ may now be established.

2) The unknown two-port is now inserted and the standing wave pattern is recorded for different values of θ. The reflection coefficient at the reference plane 2 is then plotted with the aid of a polar chart.

Data reduction proceeds in the following manner: let $\theta_0 = \arg a_1 - \arg a_2$ be the angle corresponding to position zero on the dial of the calaibrated phase shifter. Equation (7) then becomes

$$K_2 = S_{22} + S_{21}\left|\frac{a_1}{a_2}\right| e^{j(\theta_0 \pm |\theta|)} \qquad (8)$$

where $|\theta|$ is the reading on the phase shifter. Assume that l is the abscissa of the first minimum of the short-circuit pattern recorded under (1) (Fig. 1). At this minimum

$$(\arg b_2)_l - (\arg a_2)_l = \pi \qquad (9)$$

or,

$$(\arg b_2)_0 - \beta l - (\arg a_2)_0 - \beta l = \pi \qquad (10)$$

where β is the propagation constant in the waveguide. Omitting the subscript $_0$ for reference plane 2, and because

$$\arg b_2 = \arg a_1 - \beta d \qquad (11)$$

we find

$$\theta_0 = \arg a_1 - \arg a_2 = \pi + 2\beta l + \beta d. \qquad (12)$$

It may be easily checked that a positive phase shift of a_1 moves the minimum in the sense of increasing l. The phase angle of S_{21} is found at the point along the circle where $|\theta| = \mp\theta_0$. The appropriate sign is readily deduced from the above mentioned property. Typical results are shown for a Philips PP4422X isolator (frequency range 8.5–9.6 GHz), measured at 10 GHz (Fig. 4). Here the 1–2 direction corresponds to the forward direction of the isolator. In this case, the initial circle has been compressed by appropriate settings of attenuator A_1, thus reducing the value of $|a_1/a_2|$. Four values of $|S_{21}|$ have been found:

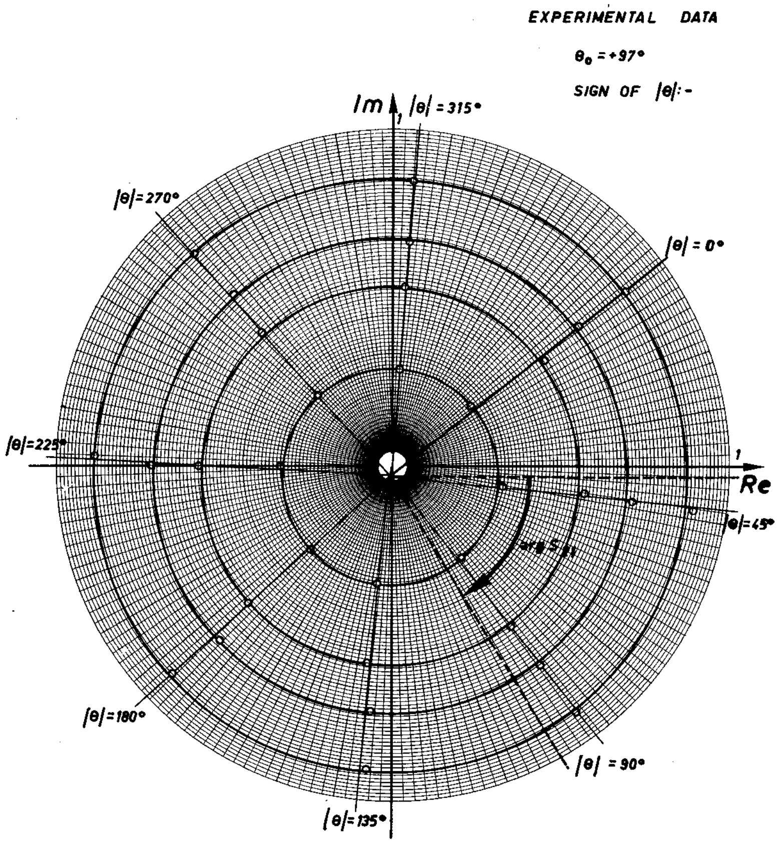

Fig. 4. Plot of the measured reflection coefficients in the complex plane.

Attenuator setting (dB)	$\lvert S_{21}\rvert$
0	0.884
2	0.870
4	0.886
9	0.896.

Averaging yields: $|S_{21}| = 0.884$. From the experimental data listed in the figure it is also possible to deduce the value of arg S_{21} which is $-52°$.

Only the straight line with $|\theta| = 0$ is important, because it fixes the origin for the phase determination. The slight deformation of the star with center S_{22} is due to the inaccuracy of the phase shifter. It is meaningless, however, because in the complex plane we deal with real phase difference angles and not with their indications on the phase shifter.

Careful expansion of the center of the chart allows accurate determination of S_{22}. Parameters S_{12} and S_{11} are found by turning the structure end for end, and applying the same procedure as above. If the test structure happens to have a small S_{12}, e.g., of the order of S_{11} (as would be the case with isolators), the original circle must be expanded by reducing $|a_2|$. In case of a very high insertion loss, some difficulties in recording the standing wave pattern may be encountered because the power level in the slotted waveguide drops when one tries to expand the circle too much. However, it is still possible to measure an insertion loss of some 40 dB, and this with a better accuracy than with the former methods.

A theoretical study of the experimental error has been performed, but is too elaborate to be given in any detail. The theory shows that accuracies of one percent on the magnitude and 2° on the phase angle are easily achieved with run of the mill microwave equipment.

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Comments on "A Technique for Measuring Individual Modes Propagating in Overmoded Waveguide"

In a recent paper, Levinson and Rubinstein[1] presented a technique for measuring individual modes propagating in overmoded waveguide. Although the measurement technique described therein is practical and effective, one of the conclusions drawn from the results appears to be misleading. The measurement consists of sampling the voltage amplitude and relative phase of a propagating multimode 7 GHz signal by means of a series of probes located about the walls of a section of WR 650 waveguide. From this voltage data and the appropriate calculations, the relative power propagating in each mode through the WR 650 waveguide is determined. One advantage of measuring in oversized waveguide is that the modes of interest are far from cutoff for the tests performed at 7 GHz. According to Taub[2] a maximum error of only six percent can occur in the calculations of relative power if, for simplicity, free-space impedance is assumed instead of the individual wave impedance for each mode. As the paper correctly indicates, the relative power ratios determined for the modes propagating in the oversized (WR 650) waveguide are the same as the relative power ratios of the modes propagating in the standard (WR 284) waveguide, assuming very little mode conversion caused by the connecting tapered section.

The misleading conclusion implied in the paper (Fig. 10 and associated text), however, is that the relative *amplitude* ratios for the modes propagating in the oversized waveguide are the same as the amplitude ratios for the respective modes propagating in the standard waveguide. This conclusion is not correct because the ratio of the mode wave impedances are different in the two different waveguide sizes. Hence, as two modes having a constant power ratio propagate from a waveguide cross section in which one mode is near cutoff to a larger waveguide cross section in which both modes are far from cutoff, the mode amplitude ratio decreases, assuming no mode conversion, according to the square root of the appropriate wave impedances. Applying this correction will result in good agreement between the reported tests and the results of the theoretical analysis by Felson.[3]

Finally, the paper does not indicate whether the relative phase data in Figs. 6, 7, 8, and 10 is referenced at the test probes or at the particular mode-exciting device. Because of the dispersive nature of the taper and the standard size waveguide, the relative mode phases at these locations are significantly different.

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[4] Manuscript received August 23, 1966.

Author's Reply[4]

We would like to thank Puttre for his comments on our paper on mode measurements[1] and offer the following clarifications. Puttre points out that it is implied that the relative mode amplitude ratios measured in the oversize guide are the same in the standard waveguide. Also, he has raised the question as to the plane at which the reported relative phase information was referenced, and noted that any translation of this data to the mode-excited device must account for the dispersive properties of the taper and standard waveguide. Concerning this last question, all phase data given are referenced at the plane of the measurement probes. Also, since at the time this work was performed our prime interest was in specifying the over-moded aperture illumination at the plane of the large end of the taper, any translation of the measured relative phase was carried back only to that plane. Even here, although the corrections were small, we did account for the relative phase dispersion between modes to obtain the best accuracy.

With reference to the translation of the relative mode amplitudes from the oversize to standard waveguide, here again we were concerned only with specifying their measure at the radiating aperture. However, sufficient information is available from the measurements whereby the mode amplitude ratios measured in one size guide can be readily translated to the ratio in any other size guide. We feel that Puttre's point is well taken, and, for the sake of completeness, it will be shown here how the translations are accomplished. It should first be pointed out that the ratio of the amplitudes of any TE_{mn} modes does not change where the waveguides being considered have the same aspect ratio, i.e., $a_1/b_1 = a_2/b_2$. This is due to the fact that the manner in which the measurements are performed yields relative amplitudes which are proportional to the amplitudes of the electric field components only. In general, however, the amplitude ratios of modes in different size waveguides are different, and can be related through the respective cutoff frequencies for the modes in each size guide, as is shown below.

Using the notation as in Moreno,[5] and considering, for example, the translation of the ratio of two TE_{mn} modes, we get

$$\frac{A_{TE_{mn1}}}{A_{TE_{mn2}}} \propto \frac{B_{mn1}K^2_{mn1}}{B_{mn2}K^2_{mn2}} \tag{1}$$

where the term on the left side is the measured amplitude ratio, and

$$K^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = \left(\frac{2\pi}{c}\right)^2 f_c^2. \tag{2}$$

Letting the bracket subscript denote waveguide sizes 1 and 2, the translation ratio is given by

$$\frac{[A_{mn1}/A_{mn2}]_1}{[A_{mn1}/A_{mn2}]_2} = \frac{[f_{cmn1}/f_{cmn2}]_1^2}{[f_{cmn1}/f_{cmn2}]_2^2}. \tag{3}$$

As noted before, for waveguides having the same aspect ratios, the ratio of the cutoff frequencies remains the same, and thus the mode amplitude ratios are the same.

In performing the above calculation, the accuracy is not dependent on how near to cutoff the respective modes are. For TE_{mn} modes, this dependency shows up only in the expressions for the magnetic field components. The converse is true when considering TM_{mn} modes, and in the following equations for translating the electric field amplitude ratios of TM_{mn} modes, the computations could blow-up if one of the modes is near cutoff in one of the guides.

For TM_{mn} modes, then, we have in general,

$$\frac{A_{TM_{mn1}}}{A_{TM_{mn2}}} \propto \frac{A_{mn1}\beta_{mn1}K^2_{mn2}}{A_{mn2}\beta_{mn2}K^2_{mn1}} \tag{4}$$

where

$$\beta^2 = \frac{\omega^2}{c^2} - K^2 = \left(\frac{2\pi}{c}\right)^2(f^2 - f_c^2). \tag{5}$$

The translation ratio for the two different size guides is then,

$$\frac{[A_{mn1}/A_{mn2}]_1}{[A_{mn1}/A_{mn2}]_2} = \frac{[f_{cmn2}/f_{cmn1}]_1^2}{[f_{cmn2}/f_{cmn1}]_2^2} \cdot \left\{\frac{[f^2 - f^2_{cmn1}/f^2 - f^2_{cmn2}]_1}{[f^2 - f^2_{cmn1}/f^2 - f^2_{cmn2}]_2}\right\}^{1/2}. \tag{6}$$

For a combination of TE_{mn} and TM_{mn} modes, the translation ratio is,

[5] T. Moreno, *Microwave Transmission Design Data*. New York: Dover, 1958, p. 115.